ON THE REGULATORS OF REAL QUADRATIC NUMBER FIELDS

PARK, JEONGHO

Abstract. In this paper the abundance of real quadratic number fields with regulators of the size $\gg \log^2 D$ is proved, examining the distribution of quadratic integers of small norms. This is true for almost all fundamental discriminants in some sense.

Introduction

It is believed that in most cases the real quadratic number field $\mathbb{Q}(\sqrt{d})$ has large fundamental unit ε_d compared to its discriminant D. Together with the class number formula of Dirichlet, very convincing quantitative conjectures on the distribution of class numbers such as Hooley's conjecture also suggest that in such cases the size of regulator $\log \varepsilon_d$ shall be as large as $\frac{\sqrt{d}}{\log^2 d}$. In the case of fundamental discriminants D, it is proved that there exists $c > 1$ such that the inequality $\varepsilon_d > \exp(\log^c D)$ is true for infinitely many D's: see [17], [4]. Concerning the density of d's with large fundamental unit, it is known that for almost every D one has $\varepsilon_d > D^{\frac{7}{4}-\epsilon}$ where $\epsilon > 0$ is arbitrary (see the introduction of [6] and corollary 1 of [7]), and it is also proved by the same authors that $\varepsilon_d > D^{3-\epsilon}$ holds for a positive density of D's. Concerning the density of D's with large regulator, the results up to now are basically dealing with $\log \varepsilon_d \gg \log D$.

This paper treats the abundance of regulators that are not too small, based on the interplay of continued fraction and the distribution of quadratic integers of given norm. In spite of its elementary flavor, the implication is a bit new and leads us to further researches in another direction. We denote by $\omega(n)$ the number of distinct prime factors of n.

Theorem 0.0.1. *Let μ be a square-free integer. For each positive integer T, let ξ_T be a root of $X^2 - TX + \mu = 0$ and D the discriminant of $\mathbb{Q}(\xi_T)$*

Date: July 12, 2012.

1991 *Mathematics Subject Classification.* 11R29, 11Y40, 11J68.

Key words and phrases. Real quadratic number field, Class number, Fundamental unit, regulator.

Supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (2010-0026473).

when ξ_T is irrational. Let d be the maximal square-free factor of D. Then (i)*there exists a constant C such that for almost all positive integer T*

$$\log \varepsilon_d > \frac{1}{\log \mu}\left(\log \frac{\sqrt{D}}{2}\right)^2 - \left(3 - \frac{2\log 2}{\log \mu}\right)\log \frac{\sqrt{D}}{2} - C$$

Moreover, let $f(N)$ be the number of distinct fields in $\{\mathbb{Q}(\xi_T) \mid 1 < T < N\}$. Then (ii)

$$\liminf_{N\to\infty} \frac{f(N)}{N} \geq 2^{-\omega(\mu)}$$

The costant C in the theorem can be made effective; note that this improves the effective bound in [4] in case $n = 1$. It is obvious that an improved bound can be obtained for general n's using lemma 2.0.10.

1. Preliminaries

Some of the classical results will be helpful to give a site of what has been known so far. Let d be a positive square-free integer and D the field discriminant of $\mathbb{Q}(\sqrt{d})$. The notations $L(s,\chi)$, ε_d, h_d represent the L-function, fundamental unit, and class number of $\mathbb{Q}(\sqrt{d})$ as usual. The followings are fairly well known.

Proposition 1.0.2 (Dirichlet)**.**

$$h_d = \frac{\sqrt{D}L(1,\chi)}{2\log \varepsilon_d}$$

Proposition 1.0.3.

$$\frac{1}{D^\epsilon} \ll L(1,\chi) \ll \log D$$

Under GRH this can be strengthened to

$$\frac{1}{\log\log D} \ll L(1,\chi) \ll \log\log D$$

(See the introduction of [16] and its references.)

This implies that the variation of $L(1,\chi)$ is negligible compared to that of $\log \varepsilon_d$. In the same philosophy, the asymptotic behavior of $h_d \log \varepsilon_d$ has been known to a precision as indicated below.

Proposition 1.0.4 ([3], [11])**.**

$$\log\left(h_d \log \varepsilon_d\right) \sim \log \sqrt{D} \quad \textit{as } D \to \infty$$

Proposition 1.0.5 ([9], [5], [1], [14], [13]).

$$\sum_{0<D<x} h_d \log \varepsilon_d \sim \frac{c}{6} x^{3/2} \quad \textit{as } x \to \infty,$$

where c is the Artin constant given by

$$c = \prod_p \left(1 - \frac{1}{p^2(p+1)}\right) = 0.8815138397\cdots$$

It is natural therefore to try to separate h_d and $\log \varepsilon_d$ in these asymptotic estimations, but it is only a hope at this moment. The most convincing and precise conjecture about the distribution of class numbers is suggested by Hooley who gave a series of conjectures about indefinite quadratic forms in [2].

Conjecture 1.0.6 (Hooley).

$$\sum_{0<D<4x,\, 4|D} h_d \sim \frac{25}{12\pi^2} x (\log x)^2$$

The other assertion of Hooley's conjecture is that in most cases, as expected, the regulator $\log \varepsilon_d$ is as large as $\sqrt{d}/\log^2 d$. It may be said that h_d constitutes the algebraic side of $\mathbb{Q}(\sqrt{d})$, whereas the analytic side of $\mathbb{Q}(\sqrt{d})$ would have ε_d as its main block. It is a sort of misfortune at this time that the theory of Diophantine approximation is essentially the only ingredient for the latter part.

Some of the notations need to be specified for the sequels. Throughout this paper μ represents a rational integer greater than 1 that is comparatively small, and d a square-free positive integer which is usually considered to be large. Let $K_d = \mathbb{Q}(\sqrt{d})$ and O_d the ring of integers of K_d, D the discriminant of K_d, and ω_d the standard basis of K_d, namely

$$\omega_d = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \bmod 4, \\ \sqrt{d} & \text{otherwise.} \end{cases}$$

Let $\omega_d = [a_0, a_1, a_2, \cdots]$ be the simple continued fraction expansion of ω_d, l the period of ω_d, $\frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n]$ a convergent, $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \cdots]$ the (n+1)-th total quotient, and $N(x)$ the usual norm of $x \in \mathbb{Q}(\sqrt{d})$. Put

$$\xi_n = \begin{cases} p_n - q_n + q_n\omega_d & \text{if } d \equiv 1 \bmod 4 \\ p_n + q_n\omega_d & \text{otherwise} \end{cases}$$

and let $\nu_n = |N(\xi_n)|$. We say that a quadratic integer ξ *comes from* a convergent to ω_d when ξ is of this form.

Recall that a quadratic irrational α is *reduced* if $\alpha > 1$ and $-1 < \overline{\alpha} < 0$. It is a classic that the continued fraction expansion of x is purely periodic if and only if x is reduced, and in particular $\sqrt{d} + \lfloor\sqrt{d}\rfloor$ is reduced so one can write $\sqrt{d} = [a_0, \overline{a_1, \cdots, 2a_0}]$. It will be shown later in a general form that $\varepsilon_d = \xi_{l-1}$.

2. Reduced ideals and the convergents to ω_d

Following the literature of [12], [8] and [15], in [17] an explicit correspondence between the set of reduced ideals of $\mathbb{Q}(\sqrt{d})$ and the set of reduced quadratic irrationals with discriminant D was given. For the sake of references in following sections, a short material in [17] is included here.

Let D be the field discriminant of K_d. Put $\omega = \frac{D+\sqrt{D}}{2}$, then 1 and ω form a $\mathbb{Z}$-basis of O_d. For $\beta_1,\ \beta_2,\ \cdots, \beta_n \in K_d$, we denote by $[\beta_1, \cdots, \beta_n]$ and by $(\beta_1, \cdots, \beta_n)$ respectively the modules in K_d generated by the elements over $\mathbb{Z}$ and over O_d. So $O_d = [1, \omega] = (1)$. Every integral ideal I has the (unique) canonical basis of the following form: $I = [a, b + c\omega]$ where $a, b, c \in \mathbb{Z}$ satisfying (i) $a > 0$, $c > 0$ and $ac = N(I)$, (ii) $a \equiv b \equiv 0 \pmod{c}$ and $N(b + c\omega) \equiv 0 \pmod{ac}$ and (iii) $-a < b + c\overline{\omega} < 0$. Then we define $\alpha(I)$ by

$$\alpha(I) = \frac{b + c\omega}{a}$$

and call $\alpha(I)$ *the quadratic irrational associated with the ideal* I. An integral ideal I is called *reduced* if $c = 1$ and $\alpha(I)$ is a reduced quadratic irrational.

Lemma 2.0.7.

$$\alpha((\xi_n)) = \alpha_{n+1}$$

Proof. Suppose $d \equiv 2$ or $3 \pmod 4$ so $\xi_n = p_n + q_n\sqrt{d}$. We have $\frac{p_n\alpha_{n+1}+p_{n-1}}{q_n\alpha_{n+1}+q_{n-1}} = \sqrt{d}$, and hence

$$\begin{aligned}
\alpha_{n+1} &= \frac{-p_{n-1} + q_{n-1}\sqrt{d}}{p_n - q_n\sqrt{d}} \\
&= \frac{-p_np_{n-1} + q_nq_{n-1}d + (p_nq_{n-1} - p_{n-1}q_n)\sqrt{d}}{\nu_n} \\
&= \frac{-(p_np_{n-1} - q_nq_{n-1}d) + (-1)^{n-1}\sqrt{d}}{\nu_n}
\end{aligned}$$

Under the correspondence specified above,

$$\begin{aligned}\alpha_{n+1} \longleftrightarrow & [\nu_n, (-1)^n(p_np_{n-1} - q_nq_{n-1}d) + \sqrt{d}] \\ &= [(p_n - q_n\sqrt{d})(p_n + q_n\sqrt{d}), (-p_{n-1} + q_{n-1}\sqrt{d})(p_n + q_n\sqrt{d})] \\ &= (p_n + q_n\sqrt{d}) \\ &= (\xi_n).\end{aligned}$$

The proof is the same in the case $d \equiv 1 \pmod 4$, with only trifling changes as $\xi_n = p_n - q_n + q_n\omega_d$, $\frac{p_n\alpha_{n+1}+p_{n-1}}{q_n\alpha_{n+1}+q_{n-1}} = \omega_d$, and $\nu_n = (p_n - q_n + q_n\omega_d)(p_n - q_n\omega_d)$. □

Lemma 2.0.8.

$$\prod_{i=1}^{l} \alpha_i = \varepsilon_d.$$

Proof. See [17]. □

Lemma 2.0.9. *An integral ideal I is reduced if (i) $N(I) < \sqrt{D}/2$ and (ii) the conjugate ideal $\overline{I}$ is relatively prime to I.*

Proof. See [17]. □

Lemma 2.0.10. *For $n \geq 0$*

$$\alpha_{n+1} = \frac{\sqrt{D}}{\nu_n} - \frac{q_{n-1}}{q_n} + O(\frac{1}{q_n^2\sqrt{D}})$$

Proof. The cases $d \equiv 2$ or $3 \pmod 4$ are easier in computation, so here we assume $d \equiv 1 \pmod 4$ and hence $\omega_d = \frac{1+\sqrt{d}}{2}$. Recall that the continued fraction expansion of ω_d has a natural geometric interpretation on xy-plane. Let O be the origin, $A = (q_{n-1}, p_{n-1})$, $B = (q_n, p_n)$, C the intersection of $\overline{AB}$ and the line $y = \omega_d x$, and $D = (q_n, \omega_d q_n)$. Then $[\overline{AC} : \overline{CB}] = [\alpha_{n+1} : 1]$ and the area of $\triangle OAB$ is $1/2$. Observe that the area of $\triangle OBD$ is $\frac{1}{2}|(p_n - q_n\omega_d)q_n|$.

We have

$$\frac{\xi_n\overline{\xi_n}}{q_n^2} = \left(\frac{p_n}{q_n} - 1 + \omega_d\right)\left(\frac{p_n}{q_n} - 1 + 1 - \omega_d\right) = \pm\frac{\nu_n}{q_n^2}$$

or

$$\frac{p_n}{q_n} - \omega_d = \frac{\pm\nu_n}{q_n(p_n - q_n + \omega_d q_n)}$$

and therefore

$$\begin{aligned}
|(p_n - q_n\omega_d)q_n| &= \frac{\nu_n}{p_n/q_n - 1 + \omega_d} \\
&= \frac{\nu_n}{2\omega_d - 1 \pm \frac{\nu_n}{q_n(p_n - q_n + \omega_d q_n)}} \\
&= \frac{\nu_n}{2\omega_d - 1}\left(\frac{1}{1 \pm \frac{\nu_n}{(2\omega_d - 1)q_n(p_n - q_n + \omega_d q_n)}}\right) \\
&= \frac{\nu_n}{\sqrt{d}}\left(1 + O\left(\frac{\nu_n}{q_n^2 d}\right)\right)
\end{aligned}$$

It follows that the area of $\triangle BCD$ is $\frac{1-q_{n-1}/q_n}{1+\alpha_{n+1}}\frac{\nu_n}{2\sqrt{d}}\left(1 + O\left(\frac{\nu_n}{q_n^2 d}\right)\right)$, and hence

$$\begin{aligned}
|\triangle OBC| &= \left(1 - \frac{1 - q_{n-1}/q_n}{1 + \alpha_{n+1}}\right)\frac{\nu_n}{2\sqrt{d}}\left(1 + O\left(\frac{\nu_n}{q_n^2 d}\right)\right) \\
&= \left(\frac{\alpha_{n+1} + q_{n-1}/q_n}{1 + \alpha_{n+1}}\right)\frac{\nu_n}{2\sqrt{d}}\left(1 + O\left(\frac{\nu_n}{q_n^2 d}\right)\right)
\end{aligned}$$

But $|\triangle OBC| = |\triangle OAB| \cdot \frac{1}{1+\alpha_{n+1}} = \frac{1}{2(1+\alpha_{n+1})}$, which proves the lemma in case $d \equiv 1 \pmod 4$. When $d \equiv 2$ or $3 \pmod 4$, exactly the same computation with continued fraction of $\omega_d = \sqrt{d}$ completes the proof. $\square$

Note that any quadratic integer ξ with square-free norm μ gives an integral ideal which is relatively prime to its conjugate; therefore the conditions of lemma 2.0.9 are satisfied by ξ^m if $|N(\xi^m)| \leq \omega_d - 1$. Based on these lemmas, the problem of fundamental units is naturally translated to the problem of quadratic integers of small norms.

3. Quadratic integers of small norms

Assume $x = a + b\omega_d \in O_d$ where a, b are positive integers that are relatively prime and $|N(x)| = \mu < \omega_d - 1$. It easily follows that $\frac{a+b}{b} - \omega_d < \frac{1}{2b^2}$ (or $\frac{a}{b} - \omega_d < \frac{1}{2b^2}$), which implies $\frac{a+b}{b}$ (or $\frac{a}{b}$) is a convergent to ω_d. Recall that for every positive integer μ there are only finitely many non-associated integers in O_d of norm $\pm\mu$. As the unit rank of O_d is 1, in each class of these associated integers one can choose the least element among those irrational ones greater than 1. Let $F_{(d,\mu)} = \{\xi_1, \cdots, \xi_t\}$ be the set of these least elements, and define $E_\mu(x) = \left|\mathbb{R}_{>1}^{<x} \bigcap \left(\bigcup_{d:squarefree} F_{(d,\mu)}\right)\right|$.

Proposition 3.0.11. *If $\omega_d - 1 > \mu \geq 1$ and μ is square-free, then the elements of $F_{(d,\mu)}$ are of the form $a + b\omega_d$ where $\frac{a+b}{b}$ (or $\frac{a}{b}$) $= [a_0, a_1, \cdots, a_n]$ and $n < l$, $\omega_d = [a_0, \overline{a_1, \cdots, a_l}]$.*

Proof. Clear from the best approximation property of continued fraction. □

Remark. The assertion $\varepsilon_d = \xi_{l-1}$ easily follows from this proposition and Lemma 2.0.10.

Theorem 3.0.12. *Let $\mu < M < x$. Then*
(i) $E_\mu(x) < 2x - 2\sqrt{\mu} + O(1)$
(ii) $E_\mu(x) > 2\left(1 - \frac{1}{2M-1}\right)x - \left(\sum_{\omega_d < M} \frac{|F_{(d,\mu)}|}{\log \varepsilon_d}\right) \log x + O(1)$

Proof. Observe that every quadratic integer y of norm $\pm\mu$ is a solution of the equation $X^2 + mX \pm \mu = 0$ for some $m \in \mathbb{Z}$. The number of real quadratic integers y greater than 1 with trace m and norm $\pm\mu$ is 2 if $m > 2\sqrt{\mu}$ and 1 if $m \leq 2\sqrt{\mu}$. With the expression $\xi = \frac{m+\sqrt{m^2 \pm 4\mu}}{2}$ the first inequality is of triviality. As for the second inequality, note that such y must be of the form $\xi\varepsilon_d^k$ for some $\xi \in F_{(d,\mu)}$ and $k \geq 0$. $E_\mu(x)$ counts the numbers with $k = 0$, so we can simply exclude the numbers $\xi\varepsilon_d^k$ less than x with $k \geq 1$. But $\xi\varepsilon_d^k < x$ if and only if $k < \frac{\log x - \log \xi}{\log \varepsilon_d}$, and therefore

$$\sharp\left\{\xi\varepsilon_d^k < x \ \mid \ k \geq 1,\ \xi \in F_{(d,\mu)},\ \omega_d \leq M\right\} < \left(\sum_{\omega_d \leq M} \frac{|F_{(d,\mu)}|}{\log \varepsilon_d}\right) \log x.$$

Now consider $\omega_d > M$ and write $\xi\varepsilon_d^k < x \iff \xi < x$ and $\varepsilon_d^k < \frac{x}{\xi}$. Since $\omega_d > M > \mu$, as mentioned at the beginning of this section $\xi = n\acute{\xi}$ for some $n \in \mathbb{N}$ where $\acute{\xi}$ comes from a convergent to ω_d and hence $\xi \geq 2M - 1$. Therefore the contribution to $E_\mu(x)$ from $\omega_d > M$ and $k \geq 1$ is less than the number of quadratic units in the interval $(1, \frac{x}{2M-1})$, which is $\frac{2}{2M-1}x + O(1)$. □

Let $a_{(d,\mu)}$ be the set of reduced integral ideals of norm μ in O_d, so $|F_{(d,\mu)}| \leq |a_{(d,\mu)}|$.

Proposition 3.0.13. *Assume $\omega_d > \mu$ where μ is square-free. Let $\mu_1 = (\mu, 2d)$ and write $\mu = \mu_1\mu_2$. Then*

$$|a_{(d,\mu)}| = \begin{cases} 2^{\omega(\mu_2)} & \textit{if } d \textit{ is a square modulo } \mu,\ d \equiv 2 \textit{ or } 3 \pmod 4 \\ 2^{\omega(\mu_2)} & \textit{if } d \textit{ is a square modulo } \mu,\ d \equiv 1 \pmod 4,\ \mu \textit{ is odd} \\ 2^{\omega(\mu_2)+1} & \textit{if } d \textit{ is a square modulo } \mu,\ d \equiv 1 \pmod 8,\ \mu \textit{ is even} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let $I \in a_{(d,\mu)}$ and write $\alpha(I) = \frac{b+c\omega}{\mu}$. Then I is reduced if and only if $c = 1$ and $\alpha(I) > 1$, $-1 < \overline{\alpha(I)} < 0$.

Suppose $d \equiv 2$ or $3 \pmod 4$ so that $\alpha(I) = \frac{b+2d+\sqrt{d}}{\mu}$. The condition $N(b+2d+\sqrt{d}) \equiv 0 \pmod \mu$ implies $(b+2d)^2 \equiv d \pmod \mu$, so we can write $b \equiv -2d + \zeta \pmod \mu$ where $\zeta^2 \equiv d \pmod \mu$. Hence if d is not a square modulo μ there is no such ideal. The condition $-\mu < b + c\varpi < 0$ implies b varies in a complete system of residues modulo μ; hence if d is a square modulo μ, the number of possible b's is the number of solutions to $\zeta^2 \equiv d \pmod \mu$. For an odd prime factor p of μ, the congruence $\zeta^2 \equiv d \pmod p$ has two roots if $p \nmid d$ and one root if $p \mid d$. The first case easily follows from this.

Now suppose $d \equiv 1 \pmod 4$. Then $\alpha(I) = \frac{b+d/2+\sqrt{d}/2}{\mu}$ and in the same way as above we get $b^2 + db + d(d-1)/4 \equiv 0 \pmod \mu$ where b varies in a complete system of residues modulo μ. When μ is odd, 2 is a unit modulo μ so one can write $b \equiv -\frac{d}{2} + \zeta \pmod \mu$ where $\zeta^2 \equiv \frac{d}{4} \pmod \mu$. This proves the second case. When μ is even, write $\mu = 2\nu$ and consider $b^2+db+d(d-1)/4 \equiv 0 \pmod 2$ and $b^2 + db + d(d-1)/4 \equiv 0 \pmod \nu$ separately. The latter has $2^{\omega(\mu_2)}$ solutions. The former has no solution when $d \equiv 5 \pmod 8$ and two solutions when $d \equiv 1 \pmod 8$, which proves the third case. □

4. Proof of Theorem 0.0.1

Now Theorem 0.0.1 can be proved easily in the philosophy of Theorem3.1 in [17].

Proof. Let $\xi \in F_{(d,\mu)}$ where d is sufficiently large, and put $L = \lfloor \log_\mu(\sqrt{D}/2) \rfloor$. The ideals (ξ^e) and $(\overline{\xi}^e)$ of O_d are reduced if $\mu^e < \sqrt{D}/2$ by lemma 2.0.9. Now by lemma 2.0.7, 2.0.8 and 2.0.10 we have

$$\begin{aligned}
\varepsilon_d &= \prod_{i=1}^{l} \alpha_i \\
&\geq \prod_{e=1}^{L} \alpha((\xi^e)) \prod_{e=1}^{L} \alpha((\overline{\xi}^e)) \\
&= \prod_{e=1}^{L} \left(\frac{\sqrt{D}}{\mu^e} - \frac{q_{n-1}}{q_n} + O(\frac{1}{q_n^2\sqrt{d}}) \right)^2 \\
&> \prod_{e=1}^{L} \left(\frac{\sqrt{D}}{\mu^e} - 1 \right)^2
\end{aligned}$$

Taking logarithm,

$$\begin{aligned}\log \varepsilon_d &> \sum_{e=1}^{L} 2\log(\frac{\sqrt{D}}{\mu^e}-1)\\ &> 2\sum_{e=1}^{L}\left(\log\sqrt{D}-e\log\mu-\left(\frac{\sqrt{D}}{\mu^e}-1\right)^{-1}\right)\end{aligned}$$

The last term can be written

$$\begin{aligned}\sum_{e=1}^{L}\left(\frac{\sqrt{D}}{\mu^e}-1\right)^{-1} &= \sum_{e=1}^{L}\frac{\mu^e}{\sqrt{D}}\left(\frac{1}{1-\frac{\mu^e}{\sqrt{D}}}\right)\\ &= \sum_{e=1}^{L}\frac{\mu^e}{\sqrt{D}}\left(1+O(\frac{\mu^e}{\sqrt{D}})\right)\\ &= \frac{\mu}{\sqrt{D}}\left(\frac{1-\mu^L}{1-\mu}\right)+O\left(\frac{\mu^2}{D}\frac{1-\mu^{2L}}{1-\mu^2}\right)\\ &= O(1)\end{aligned}$$

and therefore

$$\begin{aligned}\log \varepsilon_d &> 2L\log\sqrt{D}-L(L+1)\log\mu-O(1)\\ &> 2\left(\frac{\log(\sqrt{D}/2)}{\log\mu}-1\right)\log\sqrt{D}-\log(\sqrt{D}/2)\left(\frac{\log(\sqrt{D}/2)}{\log\mu}+1\right)-O(1)\\ &= \frac{1}{\log\mu}\left(\log\frac{\sqrt{D}}{2}\right)^2-\left(3-\frac{2\log 2}{\log\mu}\right)\log\frac{\sqrt{D}}{2}-O(1)\end{aligned}$$

The assertion of 'almost all' in (i) is immediate from Theorem 3.0.12. (ii) follows from Theorem 3.0.12 and Proposition 3.0.13 at once too. □

5. Further researches

It is natural and traditional to order the real quadratic number fields according to their discriminants and consider the density of fundamental discriminants D with large regulators based on that order. The translation of the problem about fundamental units to one about quadratic integers of small norms may seem to have an obstacle for this purpose, but by reformulating the problem in the philosophy of [10] a connection can be found.

For $1 \leq \mu \leq 2n$ let $N_\mu^{(n)}(X)$ be the number of integral solutions (k, x, y) of

$$ky^2 \mp 2\mu ny = x^3 \mp \mu x \tag{5.0.1}$$

satisfying $0 < x < y < X$. Simple numerical tests suggest that $N_\mu^{(n)}(X)$ is less than $N_1^{(n)}(X)$ but is not too small. This means that the square-free integers $\mu \leq 2n$ appear comparatively often among norms of reduced quadratic integers that come from convergents to ω_d for $n^2 < d < (n+1)^2$ (see [10]). It shall imply that on average each continued fraction expansion of $\sqrt{d}$ gives rise to as many as $\sqrt{d}$ distinct reduced quadratic integers so that $\log \varepsilon_d \gg \sum_{m=2}^{2n} \left(\log \frac{\sqrt{D}}{m}\right) \gg \sqrt{d}$.

References

1. B.A.Datskovsky, *A mean-value theorem for class numbers of quadratic extensions, a tribute to emil grosswald*, Number Theory and Related Analysis, ed. M. Knopp and M.Sheingorn (1993), 179–242.
2. C.Hooley, *On the pellian equation and the class number of indefinite binary quadratic forms*, J. Reine Angew. Math. **353** (1984), no. 2, 98–131.
3. C.L.Siegel, *Über die classenzahl quadratischer zahlkörper*, Acta Arith. **1** (1935), 83–86.
4. C.Reiter, *Effective lower bounds on large fundamental units of real quadratic fields*, Osaka J. Math. **22** (1985), no. 4, 755–765.
5. D.Goldfeld and J.Hoffstein, *Eisenstein series of 1/2-integral weight and the mean value of real dirichlet l-series*, Invent.Math **80** (1985), 185–208.
6. E.Fouvry and F.Jouve, *A positive density of fundamental discriminants with large regulator*, Preprint.
7. ______, *Size of regulators and consecutive square-free numbers*, Preprint.
8. E.L.Ince, *Cycles of reduced ideals in quadratic fields*, Mathematical Tables **IV** (1968).
9. H.Cohen, *A course in computational algebraic number theory*, (1993), 232–234, 262–263, 290–293.
10. JH.Park, *Notes on the principal primes of real quadratic number fields*, Preprint.
11. K.Chandrasekharan and H.Maass, *Gesammelte abhandlungen*, **I** (1966), 406–409.
12. P.G.L.Dirichlet, *Vorlesungen über zahlentheorie*, F.Vieweg & Son (1894).
13. S.R.Finch, *Artin's constant*, Mathematical constants (2003), 104–109.
14. ______, *Class number theory*, (2005).
15. T.Takagi, *Shoto seisuron kogi*, Kyoritsu (1931), (in Japanese).
16. X.Li, *Upper bounds on L-functions at the edge of the critical strip*, **4** (2010), 727–755.
17. Y.Yamamoto, *Real quadratic number fields with large fundamental units*, Osaka J. Math. **8** (1971), 261–270.

Department of Mathematics (Room 117), Pohang University of Science and Technology, San 31 Hyoja Dong, Nam-Gu, Pohang 790-784, KOREA. Tel. 82-10-3047-7793.

E-mail address: pkskng@postech.ac.kr